
Subject: Secrets FFTs revealed!!

Posted by [Peter Brooker](#) on Thu, 04 May 2000 07:00:00 GMT

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I have just been through a learning curve on FFTs. Much thanks to Alan Barnett for putting me on the right track. I think I have them figured out and now want to write a reference that captures my present level of understanding. Realize that I have learned only as much of the FFT theory as needed. My motivation is that I am going to be applying the FFT functions to modeling the effect of a finite lens size on the image. (The finite lens size will chop off the higher order frequencies). Perhaps somebody else will want to expand/improve this reference. These are only my best guesses to how everything works. Perhaps this is something for the IDL FAQ.

This reference is organized as follows:

PART#1: Relate complex expansion to real Fourier series

PART#2: IDL form of complex expansion

PART#3: Specific example.

PART#1: Relate complex expansion to real Fourier series.

Assume you have a function $f(t)$ that is periodic in t with a period T .

Then there exists coefficients a_n & b_n such that

$$f(t) = a_0 + \sum_n (a_n \cos(2\pi n t / T) + b_n \sin(2\pi n t / T)), \quad n=1,2,\dots$$

This is just the Fourier series of function with period X . Nothing new here. See eq #4, section 10.3, Advanced Engineering Mathematics, Kreyszig. This expansion though assumes $-T/2 < t < T/2$

Now consider an alternate form of the Fourier series expansion.

$$f(t) = \sum_n (A_n \exp(j 2\pi n t / T)), \quad n=0,1,2,\dots$$

In order for me to be comfortable with this expansion I need to see how this expansion relates to the expansion above. In particular, how do the complex A_n relate to the real a_n and b_n ?

Consider the following:

$$I = \sum_n (A_n \exp(j 2\pi n t / T)), \quad n=0,1,2,\dots \quad j^2 = -1$$

$$= A_0 + \sum_n (A_n (\cos(2\pi n t / T) + j \sin(2\pi n t / T))), \quad n=1,2,\dots$$

$$\text{Let } A_n = (a_n + j b_n)$$

$$I = A_0 + \sum_n ((a_n + j b_n) (\cos(2\pi n t / T) + j \sin(2\pi n t / T)))$$

$$= A_o + \sum_n (aa_n \cos() - bb_n \sin()) + j(bb_n \cos() + aa_n \sin())$$

$$\text{Real}(I) = \text{Real}(A_o) + \sum_n (aa_n \cos(2\pi n t/T) - bb_n \sin(2\pi n t/T))$$

Comparing to the first expansion we see that

$$\text{Real}(A_o) = a_o, \quad aa_n = a_n, \quad -bb_n = b_n$$

To me, this proves existence of the complex expansion. Knowing one, you can figure out the other. Part #1 is complete.

Part #2: IDL form of complex expansion

Let $f(t)$ be a periodic function with period T defined on an interval $[0, T]$.

Then there exist complex A_n such that

$$f(t) = \sum_n (A_n \exp(j 2\pi n t/T)), \quad n=0, 1, 2, \dots \quad j^2 = -1$$

Divide the interval into N sections. $t_i = i T/N$

Then,

$$\begin{aligned} f(t_i) &= \sum_n (A_n \exp(j 2\pi n t_i/T)) \\ &= \sum_n (A_n \exp(j 2\pi n i (T/N)/T)) \\ &= \sum_n (A_n \exp(j 2\pi n i/N)), \quad n=0, 1, \dots \end{aligned}$$

This is exactly what is found in the IDL manual under the section for FFT. The only difference is that t has been replaced by u and A_n has been replaced by $F(u)$. Note that the period T has dropped out. Also note that t has been replaced by $t_i = i T/N$. In order for this to happen, the interval over which t is defined must be from $[0, T]$. This is different from the definition of t being defined over the interval $[-T/2, T/2]$. Perhaps this is why $b_n = -bb_n$.

*****UNFORTUNATELY IT IS WRONG*****

What is wrong is the values of n in the sum. IDL does not use the values of $n=0, 1, 2, \dots$. IDL actually uses $n = -N/2+1, -N/2+2, \dots, -1, 0, 1, \dots, N/2$. The reason for doing this must have to do with FFT theory. Note also that the number of values of n is N .

It gets more complicated. From the manual we have

$$F(u) = 1/N \sum_x (f(x) \exp(-j 2\pi u x/N)), \quad x=0, 1, \dots, N-1$$

First thing to realize is that $F(u)$ is really F_n . Where n is an integer. This comes from the fact that $f(x)$ is periodic in x .

The manual also mentions that the "frequencies" are
 $F_0, 1/(NT), 2/(NT), \dots, 1/2T, -(N-2)/(2NT), \dots, -1/NT$

After trial and error I have determined that the value of the ns range
 for $-N/2$ to $N/2$. Furthermore, the F_n are stored in the order associated
 with the following values of n

$0, 1, 2, \dots, N/2, -(N/2-1), -(N/2-1), \dots, -1$ <== this is bizarre!!

Let $N=8$. Then $N/2=4$

The F_n would be stored in an array. The array of n values associated
 with this array would be:

$[0, 1, 2, 3, 4, -3, -2, -1]$

Part #3: Specific Example

Consider the interval $t = [0, 1]$. This choice of interval implies $T=1$.
 Let $f(t) = \sin(4\pi t)$

$f(t_i) = \sin(2\pi^2 i/N)$, $i=0, 1, \dots, N$

$$\begin{aligned} f(t_i) &= \sum_n (A_n \exp(-j^2 2\pi n^2 i/N)) , \quad n = -N/2, \dots, -1, 0, 1, \dots, N/2 \\ &= A_{N/2} \dots + A_{-2} (\cos(2\pi^2 (-2)^2 i/N) + j^2 \sin(2\pi^2 (-2)^2 i/N)) + \\ &\quad + A_0 + A_{-1} \exp() + A_1 \exp() + \\ &\quad A_2 (\cos(2\pi^2 (2)^2 i/N) + j^2 \sin(2\pi^2 (2)^2 i/N)) \\ &+ A_3 \exp() + \dots \\ &= \dots + A_{-2} \cos(2\pi^2 i/N) + A_2 \cos(2\pi^2 i/N) + \\ &\quad + A_{-2} j^2 (-1) \sin(2\pi^2 i/N) + A_2 j^2 \sin(2\pi^2 i/N) + \dots \\ &= \dots + (A_{-2} + A_2) \cos(2\pi^2 i/N) + j^2 (-A_{-2} + A_2) \sin(2\pi^2 i/N) \\ &+ \dots \end{aligned}$$

where A_{-2} stands for A_n where $n = -2$

Equating the series to $\sin(2\pi^2 i/n)$ we conclude

$A_n = 0$ for all n except $n = -2$ or $n = 2$.

$A_{-2} + A_2 = 0$
 $j^2 (-A_{-2} + A_2) = 1$

Let $A_{-2} = (a_{-2} + j^2 b_{-2})$ and $A_2 = (a_2 + j^2 b_2)$

The above equations imply

$(a_{-2} + a_2) + j^2 (b_{-2} + b_2) = 0$ &
 $j^2 [(-a_{-2} + a_2) + j^2 (-b_{-2} + b_2)] = 1$

$$\Rightarrow a_{-n2} + a_{2} = 0, b_{-n2} + b_{2} = 0 \Rightarrow a_{-n2} = -a_{n2}, b_{-n2} = -b_{n2}$$

$$\Rightarrow 2a_{-n2} = 0 \Rightarrow a_{-n2} = a_{2} = 0$$

$$\Rightarrow j*(2b_{-2}) = 1 \Rightarrow 2b_{-2} = -1/2, b_{-2} = 1/2$$

$$A_{-n2} = 0 + j*(1/2)$$

$$A_{2} = 0 + j*(-1/2)$$

We now have calculated the solutions.

The following code calculates this and displays the correct answers. It shows how to plot A_n vs n correctly.

```
;idl_program fft_sine.pro
TT=1
Npts=100
t=findgen(Npts)/(Npts-1)*TT
f_t=sin(4.*!pi*t/TT)
!p.multi=[0,2,3]
plot,t,f_t, title='f(t) vs t'
A_n=fft(f_t,-1) ; complex fourier coefficients

plot,float(A_n),yrange=[-.5,.5],title='float(A_n)'
plot,imaginary(A_n),yrange=[-.5,.5],title='imaginary(A_n)'

a=findgen(Npts/2+1)
b=-reverse(findgen(Npts/2-1)+1)
c=[a,b] ; c=[-N/2+1,-N/2+2, ..., -1,0,1,...,N/2]
print,c
sub=sort(c)
plot,c(sub),float(A_n(sub)),yrange=[-.5,.5],title='float(A_n ) vs n'
plot,c(sub),imaginary(A_n(sub)),yrange=[-.5,.5], $
    title='imaginary(A_n) vs n'
plot,c(sub),imaginary(A_n(sub)),xrange=[-5,5],$
    title='imaginary(An) vs n' ; finer x scale
end
```
