
Subject: FFT secrets revealed!! Update 6/5/2000

Posted by [Peter Brooker](#) on Mon, 05 Jun 2000 07:00:00 GMT

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<!doctype html public "-//w3c//dtd html 4.0 transitional//en">

<html>

Below is a repeat of my original post to the group in which I tried to explain FFT use

from the point of view of somebody who has little experience of FFT algorithms but

does understand the concept of Fourier series as well as Fourier transforms.

This update

includes a large modification to part 1 that develops the complex expansion from the

series expansion. In it I show how all the imaginary components

of the complex FFT expression cancel to zero. Modifications were also made to part2.

<p>This reference is organized as follows:

PART#1:Relate complex expansion to real Fourier series

PART#2:IDL form of complex expansion

PART#3:Specific example: $f(t) = \sin(4\pi t)$

PART#4:Specific example: $T(x,y) = \sin(6\pi x) + \cos(4\pi y)$

<p>PART#1: Relate complex expansion to real Fourier series.

Assume you have a function $f(t)$ that is periodic in t with a period T .

Then there exists coefficients a_n & b_n such that

<p> $f(t) = a_0 + \sum_n (a_n \cos(2\pi n t/T) + b_n \sin(2\pi n t/T))$, $n=1,2,\dots$

<p>The coefficients are given by:

<p> $a_0 = 2/T \int f(t) dt$

<p> $a_n = 2/T \int f(t) \cos(2\pi n t/T) dt$

<p> $b_n = 2/T \int f(t) \sin(2\pi n t/T) dt$

<p>The limits of the integration are from $-T/2$ to $T/2$.

<p>This is just the Fourier series of a function with period T . Nothing new

here. See eq #4, section 10.3, Advanced Engineering Mathematics,

Kreyszig. This expansion though assumes $-T/2 \leq t \leq T/2$

<p>Let's now define the complex coefficients:

<p> $c_0 = a_0$

 $c_n = 1/2 (a_n + j b_n)$

<p>Then for $n = 1,2,\dots$

<p> $c_n = 1/2 (a_n + j b_n)$

 $= 2/T \cdot 1/2 (\int f(t) \cos(2\pi n t/T) dt +$

$j \int f(t) \sin(2\pi n t/T) dt)$

 $= 1/T \int f(t) [\cos(2\pi n t/T) + j \sin(2\pi n t/T)] dt$

$$c_n = \frac{1}{T} \int_0^T f(t) \exp(j 2 \pi n t / T) dt$$

Since $\exp(0)=1$ we can write:

$c_n = \frac{1}{T} \int_0^T f(t) \exp(j 2 \pi n t / T) dt$, $n = 0, 1, 2, \dots$

Note also that $[(c_n)^*] = \frac{1}{T} \int_0^T f(t) \exp(-j 2 \pi n t / T) dt$

$$= \frac{1}{T} \int_0^T f(t) \exp(j 2 \pi (-n) t / T) dt$$

$$= c_{-n}$$

Here $[(c_n)^*]$ denotes the complex conjugate of the element inside ()

(Sorry for the clumsy notation!!!)

Now here is something completely different ...

Given the definition of c_n above:

$f(t) = \sum_n (c_n \exp(-j 2 \pi n t / T))$, $n = \dots, -2, -1, 0, 1, 2, \dots$

Proof:

Define AA and BB by:

$\sum_n (c_n \exp(-j 2 \pi n t / T)) = AA + c_0 + BB$,

where $AA = \sum_n (c_n \exp(-j 2 \pi n t / T))$, $n = \dots, -2, -1$

$$BB = \sum_n (c_n \exp(-j 2 \pi n t / T))$$
 , $n = 1, 2, \dots$

Let's work on AA some more:

$AA = \sum_n (c_n \exp(-j 2 \pi n t / T))$, $n = \dots, -2, -1$

$$= \sum_n (c_{-n} \exp(-j 2 \pi (-n) t / T))$$
 , $n = 1, 2, \dots$

Now $c_{-n} = [(c_n)^*]$ and $\exp(-j 2 \pi (-n) t / T) = [\exp(j 2 \pi n t / T)]^*$

therefore,

$$c_{-n} \exp(-j 2 \pi (-n) t / T) = [(c_n)^*] * [\exp(j 2 \pi n t / T)]^*$$

$$= [(c_n \exp(-j 2 \pi n t / T))]^*$$

Using this we have;

$AA = \sum_n [(c_n \exp(-j 2 \pi n t / T))]^*$, $n = 1, 2, \dots$

$$= [\sum_n (c_n \exp(-j 2 \pi n t / T))]^*$$

Comparing this expression to sum BB we see that $AA = (BB)^*$

Let's write the original sum again with this information:

$\sum_n (c_n \exp(-j 2 \pi n t / T)) = AA + c_0 + BB = (BB)^* + c_0 + BB$

$$= c_0 + 2 \operatorname{Re}(BB)$$

Now $BB = \sum_n (c_n \exp(-j^*u))$, $n=1,2,\dots$ and $u = 2\pi n t/T$

$$= \sum_n (1/2(a_n + j^*b_n)(\cos(u) - j^*\sin(u))$$

$$= 1/2 \sum_n (a_n \cos(u) + j^*(-j^*)b_n \sin(u) + j^*(b_n \cos(u) - a_n \sin(u))$$

$$= 1/2 \sum_n (a_n \cos(u) + b_n \sin(u) + j^*(b_n \cos(u) - a_n \sin(u))$$

From this we see that:

$$2\text{Re}(BB) = 2 \cdot 1/2 \sum_n (a_n \cos(u) + b_n \sin(u))$$

$$= \sum_n (a_n \cos(u) + b_n \sin(u))$$

Now we have:

$$\sum_n (c_n \exp(-j^*2\pi n t/T)) = c_0 + 2\text{Re}(BB)$$

$$= a_0 + \sum_n (a_n \cos(u) + b_n \sin(u)), n=1,2,\dots$$

This is exactly the definition for $f(t)$. I have therefore proved that

$$f(t) = \sum_n (c_n \exp(-j^*2\pi n t/T)), n= \dots, -2, -1, 0, 1, 2, \dots$$

Part #2: IDL form of complex expansion

Let $f(t)$ be a periodic function with period T defined on an interval $[0, T]$.

Then there exist complex A_n such that

$$f(t) = \sum_n (A_n \exp(j^*2\pi n t/T)), n= \dots, -1, 0, 1, \dots$$

$$j^*j = -1$$

Divide the interval into N sections. $t - t_i = i^*T/N$

Then,

$$f(t_i) = \sum_n (A_n \exp(j^*2\pi n t_i/T))$$

$$= \sum_n (A_n \exp(j^*2\pi n i^*(T/N)/T))$$

$$= \sum_n (A_n \exp(j^*2\pi n i/N))$$

$n= \dots, -1, 0, 1, \dots$

What about the n s?

IDL truncates the sum from $-N/2+1, -N/2+2, \dots, -1, 0, 1, \dots, N/2$

This is how IDL performs the FFT. This is found in the IDL manual under the section for FFT. The only difference is that t has been replaced by u and A_n has been replaced by $F(u)$. Note that the period T has dropped out. Also note that t has been replaced by $t_i = i^*T/N$. In order for this to happen, the interval over which t is defined must be from $[0, T]$. This is different from the definition of t being defined over the interval $[-T/2, T/2]$.

It gets more complicated. From the manual we have

$$F(u) = 1/N \sum_x (f(x) \exp(-j^*2\pi u x/N)), x=0, 1, \dots, N-1$$


```

<br>&nbsp;&nbsp;  endfor
<br>;
<br>!p.multi=[0,2,2]
<br>shade_surf,T,x,y,xtitle='x',ytitle='y',title='T(x,y)'
<br>;
<br>;
<br>C=fft(T,-1) ; complex fourier coefficients
<br>surface,float(C)
<br>aaa=where(float(c) gt .4)
<br>surface,imaginary(C)
<br>;
<br>a=findgen(Nx/2+1)
<br>b=-reverse(findgen(Nx/2-1)+1)
<br>ns=[a,b] ; this is the array of n's associated with C(n). n goes with
x
<br>;print,ns ; n goes from 0,...,Nx/2, -(Nx/2-1),...,-1
<br>subn=sort(ns) ;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;& n goes with x
<br>n_sort=ns(subn)
<br>;
<br>a=findgen(Ny/2+1)
<br>b=-reverse(findgen(Ny/2-1)+1)
<br>ms=[a,b] ; this is the array of m's associated with C(n,m)
<br>print,ms ; m goes from 0,...,Ny/2, -(Ny/2-1),...,-1
<br>subm=sort(ms) ; m goes with y
<br>m_sort=ms(subm)
<br>;
<br>sub_n_p3=where(ns eq 3)
<br>sub_n_n3=where(ns eq -3)
<br>sub_n_0=where(ns eq 0)
<br>;
<br>sub_m_p2=where(ms eq 2)
<br>sub_m_n2=where(ms eq -2)
<br>sub_m_0=where(ms eq 0)
<br>;
<br> print,'C(3,0),c(-3,0)=' ,C(sub_n_p3,sub_m_0),C(sub_n_n3,sub_m_0)
<br>;
<br> print,'C(0,2),c(0,-2)=' ,C(sub_n_0,sub_m_p2),C(sub_n_0,sub_m_n2)
<br>;
<br>; now we need to define CC(n,m) to have normal scaling in n & amp; m.
<br>;
<br>CC=C*0.
<br>;
<br>for n=0,Nx-1 do begin
<br>&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;& for m=0,Ny-1 do begin
<br>&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&&nbsp;&nbsp;& CC(subn(n),subm(m))= C(n,m)
<br>&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&&nbsp;&nbsp;& endfor
<br>&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;& endfor
<br>;

```

```
<br>surface,ABS(CC),n_sort,m_sort  
<br>;  
<br>end</html>
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