Subject: equally spaced points on a hypersphere?
Posted by robert.dimeo on Fri, 29 Oct 2004 14:51:58 GMT
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Hi,

I would like to create (n+1) equidistant points on an n-dimensional sphere. The initial information provided is the center of the sphere, the radius, and *any* point on the sphere. From that you need to find the coordinates for the remaining n points. As a simple example, three equidistant points on a 2-dimensional sphere (a circle), can be located 120 degrees apart. Any hints on how to do this in general for n-dimensions?

Thanks in advance!

Rob

P.S. This is for an extension to the Nelder-Mead downhill simplex routine, AMOEBA.PRO.

Subject: Re: equally spaced points on a hypersphere?
Posted by Matt Feinstein on Fri, 29 Oct 2004 15:16:42 GMT
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On 29 Oct 2004 07:51:58 -0700, robert.dimeo@nist.gov (Rob Dimeo) wrote:

> Hi,
>
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> sphere. The initial information provided is the center of the sphere,
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> three equidistant points on a 2-dimensional sphere (a circle), can be
> located 120 degrees apart. Any hints on how to do this in general for
> n-dimensions?

Unfortunately, when you go to dimension greater than two, there are constraints on the number of 'equidistant' points you can have on a sphere. For example, in 3-D, there are (only) five regular polyhedra, so n can only have the values 4, 6, 8, 12, and 20 for a tetrahedron, octahedron, cube, icosahedron, and dodecahedron.

Matt Feinstein

--

There is no virtue in believing something that can be proved to be true.

Subject: Re: equally spaced points on a hypersphere?
Posted by Craig Markwardt on Fri, 29 Oct 2004 15:29:31 GMT
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Matt Feinstein <nospam@here.com> writes:
> On 29 Oct 2004 07:51:58 -0700, robert.dimeo@nist.gov (Rob Dimeo)
> wrote:
>
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This is commonly called "tesselating" the sphere, or hypersphere in
this case.

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> octahedron, cube, icosahedron, and dodecahedron.

So is there any requirement that the tesselation produce a regular
polyhedron?

Clearly it is possible to place *any* number of equidistant points on
a sphere via an iterative approach. As discussed on line, start
with random placement of points, allow the points to repel each other,
iterate until you reach the lowest energy configure.

Whether such an approach will work for Rob, I don't know.

Craig

--
 -- --------------
Craig B. Markwardt, Ph.D.   EMAIL: craigmnet@REMOVEcow.physics.wisc.edu
Astrophysics, IDL, Finance, Derivatives | Remove "net" for better response
 -- --------------

Subject: Re: equally spaced points on a hypersphere?
Posted by Matt Feinstein on Fri, 29 Oct 2004 15:53:13 GMT
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On 29 Oct 2004 10:29:31 -0500, Craig Markwardt
<craigmnet@REMOVEcow.physics.wisc.edu> wrote:

> So is there any requirement that the tesselation produce a regular
> polyhedron?
>
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>

I think that if 'equidistant' means that each point has the same relation to -every- neighboring point, then it implies that the points lie on a regular polyhedron. In any case, a lowest energy configuration may only be a local minimum with respect to small variations of the positions of the points, so the global properties of such a minimum are not necessarily unique.

Matt Feinstein

--
There is no virtue in believing something that can be proved to be true.

Subject: Re: equally spaced points on a hypersphere?
Posted by tam on Fri, 29 Oct 2004 16:10:17 GMT
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Craig Markwardt wrote:
> Matt Feinstein <nospam@here.com> writes:
>
>> On 29 Oct 2004 07:51:58 -0700, robert.dimeo@nist.gov (Rob Dimeo)
>> wrote:
>>
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> with random placement of points, allow the points to repel each other,
> iterate until you reach the lowest energy configure.
>
> Whether such an approach will work for Rob, I don't know.
>
> Craig
>

I'm not sure what it means to have 'equidistant' points on a sphere.
I don't think the OP wants each point to be equidistant from all
other points -- I don't think that's possible for more than n+1 points
in an n-dimensional space.

Craig indicates one take on the problem, but the OP may want to
frame it more carefully, e.g., a different criterion might
be to maximize the mininum distance between any two points.
I don't know if that has the same solution. Matt points out
that only in a special cases will the soution be regular, for
most sets of points the 'facets' defined by points will not
all regular, equal polygons.

A quick Google search came up with
http://nrich.maths.org/askedNRICH/edited/1125.html
that gives some interesting references.

Regards,
Tom McGlynn

Subject: Re: equally spaced points on a hypersphere?
Posted by Craig Markwardt on Fri, 29 Oct 2004 16:40:11 GMT
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Matt Feinstein <nospam@here.com> writes:
>
> I think that if 'equidistant' means that each point has the same
> relation to -every- neighboring point, then it implies that the points
> lie on a regular polyhedron.

Hmm, but consider a soccer ball (truncated icosahedron). The faces
are not all regular, and yet the nearest neighbors are all
equidistant, no?

> In any case, a lowest energy
> configuration may only be a local minimum with respect to small
> variations of the positions of the points, so the global properties of
> such a minimum are not necessarily unique.

I think if one uses a 1/r^2 potential, then there is a single global
minimum. I guess it's possible for the iterator program to get stuck
elsewhere.

Craig

--
-- --------------
Craig B. Markwardt, Ph.D. EMAIL: craigmnet@REMOVEcow.physics.wisc.edu
Astrophysics, IDL, Finance, Derivatives | Remove "net" for better response
-- --------------

Subject: Re: equally spaced points on a hypersphere?
Posted by James Kuyper on Fri, 29 Oct 2004 16:52:08 GMT
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Craig Markwardt wrote:
> Matt Feinstein <nospam@here.com> writes:
>
>> On 29 Oct 2004 07:51:58 -0700, robert.dimeo@nist.gov (Rob Dimeo)
>> wrote:
>>
>>
>>> Hi,
>>>
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>>> sphere. The initial information provided is the center of the sphere,
>>> the radius, and *any* point on the sphere. From that you need to find

>>> the coordinates for the remaining n points. As a simple example,
>>> three equidistant points on a 2-dimensional sphere (a circle), can be
>>> located 120 degrees apart. Any hints on how to do this in general for
>>> n-dimensions?

For n=1, the solution is two points at +/-1.

For n>1, take the solution for n-1 dimensions, centered at the origin. Add one dimension, and give all those points a value of 0 in the new dimension. Add a point with a value of 'x' for the new dimension, with all it's other coordinates set to 0. Calculate the distance of the new point from any of the old points, as a function of 'x'. Solve for the value of 'x' that puts the new point at the same distance from the old point, as the old point was from all of the other points. By symmetry, the new point will also be that same distance from all of the other old points. Find the center of the new set of points. Shift all the points by the amount needed to center them on the desired location. Shift all of the points outward from the center by the same factor, to get a hypersphere of the desired radius.

Example, for n=2:

Start with two points a <-1,0> and <+1,0>.
Add a third point at <0,x>. The distance from first point is
sqrt(1+x^2). The distance between first two points is 2, so x = sqrt(3).

The center of <-1,0>, <1,0> and <0,sqrt(3)> is at <0, 1/sqrt(3)>.

Shifting all the points to center at 0, we get <-1,-1/sqrt(3)>,
<1,-sqrt(3)/3> and <0,2/sqrt(3)>. The new circle has a radius of 2/sqrt(3).

Multiply by sqrt(3)/2, to get a circle of radius one, leaving us with
<-sqrt(3)/2, -1/2>, <sqrt(3)/2, -1/2>, <0,1>.

>> Unfortunately, when you go to dimension greater than two, there are
>> constraints on the number of 'equidistant' points you can have on a
>> sphere. For example, in 3-D, there are (only) five regular polyhedra,
>> so n can only have the values 4, 6, 8, 12, and 20 for a tetrahedron,
>> octahedron, cube, icosahedron, and dodecahedron.
>
>
> So is there any requirement that the tesselation produce a regular
> polyhedron?

Yes. If all of the points are to be equidistant from each other, then the object they trace out is necessarily a regular polyhedron. I fact, there is only one polyhedron in three dimensions where all of the vertices are equidistant from each other, and that's the tetrahedron. In general, the maximum number is 1 more than the number of dimension of the sphere.

All the edges of regular polyhedra are the same length, but that's not the same thing.

> Clearly it is possible to place *any* number of equidistant points on
> a sphere via an iterative approach. As discussed on line, start
> with random placement of points, allow the points to repel each other,
> iterate until you reach the lowest energy configure.

That can't produce an equidistant set of points for any n more than 1 higher than the number of dimensions. In three dimensions it also can't produce a regular polyhedron for any n other than the ones that were listed above. Try it with 5 points on the surface of a three-dimensional sphere. The precise configuration you'll end up with depends upon the force law you use for the repulsion, but it won't be a regular polyhedron, and it certainly won't be equidistant.

Subject: Re: equally spaced points on a hypersphere?
Posted by James Kuyper on Fri, 29 Oct 2004 17:18:31 GMT

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Tom McGlynn wrote:
> Craig Markwardt wrote:
>
>> Matt Feinstein <nospam@here.com> writes:
>>
>>> On 29 Oct 2004 07:51:58 -0700, robert.dimeo@nist.gov (Rob Dimeo)
>>> wrote:
>>>
>>>
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>>>>
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> I don't think the OP wants each point to be equidistant from all
> other points -- I don't think that's possible for more than n+1 points
> in an n-dimensional space.

That's precisely what he's asking for. See above.

Subject: Re: equally spaced points on a hypersphere?

Posted by James Kuyper on Fri, 29 Oct 2004 17:24:48 GMT
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Craig Markwardt wrote:
> Matt Feinstein <nospam@here.com> writes:
>
>> I think that if 'equidistant' means that each point has the same
>> relation to -every- neighboring point, then it implies that the points
>> lie on a regular polyhedron.
>
>
> Hmm, but consider a soccer ball (truncated icosahedron). The faces
> are not all regular, and yet the nearest neighbors are all
> equidistant, no?

Nearest neighbors are equidistant, by definition. You'll never have more than one nearest neighbor, unless all of your nearest neighbors are at the same distance.

Of course, I know what you actually meant, though I can't quite figure out how to express it.

However, the original question was about a set of points which are ALL equidistant from each other. That's why the maximum is n+1, where n is the number of dimensions.

>> In any case, a lowest energy
>> configuration may only be a local minimum with respect to small
>> variations of the positions of the points, so the global properties of
>> such a minimum are not necessarily unique.
>
>
> I think if one uses a 1/r^2 potential, then there is a single global
> minimum. I guess it's possible for the iterator program to get stuck
> elsewhere.

At the very least, if you take one solution and apply an arbitrary rotation around the center of the sphere, or a reflection through an arbitrary plane passing through the center of the sphere, you will produce another solution. However, I think that there are cases with non-trivial multiple solutions, as well.

Subject: Re: equally spaced points on a hypersphere?
Posted by jeyadev on Fri, 29 Oct 2004 18:00:14 GMT
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In article <cb539436.0410290651.698f65f@posting.google.com>,

Rob Dimeo <robert.dimeo@nist.gov> wrote:
> Hi,
>
> I would like to create (n+1) equidistant points on an n-dimensional
> sphere. The initial information provided is the center of the sphere,
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> three equidistant points on a 2-dimensional sphere (a circle), can be
> located 120 degrees apart. Any hints on how to do this in general for
> n-dimensions?
>
> Thanks in advance!
>
> Rob

Munge around in sci.math.num-analysis and sci.match. Or even
alt.math.recreational. It should be in the archives. Turns up
now and then.

--

Surendar Jeyadev jeyadev1@wrc.xerox.com

Remove 1 for email address

Subject: Re: equally spaced points on a hypersphere?
Posted by robert.dimeo on Mon, 01 Nov 2004 13:34:17 GMT
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James,

Thanks very much for your solution. This is exactly what I was
looking for. The function below implements the steps outlined in your
posting.

Rob

```
function create_simplex,ndim,center = center,radius = radius
; Uses a technique suggested by James Kuyper to
; create an equilateral simplex in n-dimensions.  The return
; value is an array ndim by ndim+1.  The coordinates of
; the i-th vertex of the simplex are obtained as
; p[*,i] such as in the following call:
; IDL> p =  create_simplex(5) ; create a 5-d simplex
; IDL> vertex_3 = p[*,2]  ; the 3rd vertex in the 5-d simplex
;
p = dblarr(ndim,ndim+1)
```

```
if n_elements(center) eq 0 then center = dblarr(ndim)
if n_elements(radius) eq 0 then radius = 1d

; Fill in the values for n = 2
p[0,0] = -sqrt(3.)/2. & p[1,0] = -0.5
p[0,1] = sqrt(3.)/2. & p[1,1] = -0.5
p[1,2] = 1.
if ndim gt 2 then begin
  for j = 3,ndim do begin
    ; Solve for the value of the new dimension
    p[j-1,j] = sqrt(total((p[0:j-1,1]-p[0:j-1,0])^2)-1d)
    ; Find the center
    pc = dblarr(j)
    for i = 0,j-1 do pc[i] = (moment(p[i,0:j]))[0]
    ; Subtract the center from each of the points
    ; in the simplex so that it is centered at 0
    for k = 0,j do p[0:j-1,k] = p[0:j-1,k] - pc[0:j-1]
    ; Now find the normalization constant for unit radius
    ; of the hypersphere
    norm = sqrt(total(p[*,0]^2))
    p = temporary(p)/norm
  endfor
endif
; Scale the simplex
p = p*radius
; Shift the center
for i = 0,ndim-1 do p[i,*] = p[i,*]+center[i]
return,p
end
```

James Kuyper <kuyper@saicmodis.com> wrote in message
news:<41827538.4050709@saicmodis.com>...
>
>
> For n=1, the solution is two points at +/-1.
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> For n>1, take the solution for n-1 dimensions, centered at the origin.
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